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THE PROBLEM OF BREAKDOWN OF A VORTEX LINE*

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The conditions for the appearance of a singularity in the course of solving the quasicylindrical approximation equations for a vortex line /1/ and expanding the velocity in its neighbourhood, are studied. In /1/ the appearance of such a singularity was regarded as a signal of vortex breakdown, just as the appearance of a Goldstein singularity /2/ or of a singularity investigated in /3/ in the course of solving the boundary layer equations with a given positive pressure gradient implied the impossibility of a flow without separation. In a different approach to the study of vortex breakdown /4/ (the present level of achievement in the study of this phenomenon is elucidated in /5, 6/), the vortex flows are classified as "supercritical" if the waves can propagate with phase veloicity only in the downstream direction, and "subscritical" if the waves can propagate upstream. The vortex breakdown is regarded as a passage from the supercritical to the subcritical state at distances of the order of the vortex radius.

The circumstances under which the conditions for the appearance of a singularity and of the "criticality" are identical, are explained below. A classification of possibile singularities is given and it is shown that in the general case, when the external longitudinal pressure gradient is given, the solution near the singularity cannot, as in /2, 3/, be continued past the singularity.

1. Let us consider a vortex line with constant circulation Γ_∞ distributed along the axis of a tube of variable cross-section, in the potential flow of an incompressible fluid flowing through a tube. Such a situation is simulated in most experimental investigations for studying vortex breakdown /5/. Let us normalize the velocities and dimensions to their characteristic values in the outer flow U, L, the pressure to ρU^2 and the Reynolds number $\mathrm{Re} = UL/\nu = 1/\varepsilon^2$, $\varepsilon \to 0$. We introduce a cylindrical coordinate system (x, r) with the x axis directed along the vortex axis in the downstream direction (the flow is assumed to be axisymmetric). Let $\bar{u}, \bar{v}, \bar{w}$ be the axial, radial and asimuthal velocity component respectively. Passing to the limit in the Navier-Stokes equations, as was done in deriving the boundary layer equations $(u, v, g, y = O(1), \varepsilon \to 0)$, we obtain a system of quasicylindrical approximation equations /1, 6/ describing a slow regrouping of the internal vortex structure under the influence of viscosity

$$\begin{gathered}
\bar{u} = u(x, y) + O(\varepsilon), \quad \bar{v} = \varepsilon^2 v(x, y) r + O(\varepsilon^2) \\
\bar{w} = \varepsilon g(x, y) r + O(\varepsilon), \quad y = r^2/(2\varepsilon^2) \\
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + 2 \frac{\partial}{\partial y}\left(y \frac{\partial u}{\partial y}\right) \\
u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} = 2y \frac{\partial^2 g}{\partial y^2}, \quad \frac{\partial p}{\partial y} = \frac{g^2}{4y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
v = \sqrt{2y} \frac{\partial u}{\partial y} = 0 \text{ when } y = 0; \quad u \to u_\infty(x), \ g \to \Gamma_\infty \text{ as } y \to \infty \\
u = u_0(x_0, y), \ g = g_0(x_0, y) \text{ when } x = x_0 .
\end{gathered} \tag{1.1}$$

Here $u_\infty(x)$ is a function known from the external solution.

Various finite difference /6/ and integral /7/ methods were proposed for the numerical investigation of system (1.1). System (1.1) is non-linear parabolic, just as the system of boundary layer equations. Having the initial profiles u_0, g_0 at the points x_0 available we can, generally speaking, extend the solution regularly to the point $x_0 + \Delta x$, etc.

However, when the numerical computations are carried out, the process is frequently terminated at some point x_1 by the singularity ($v, \partial u/\partial x, \partial g/\partial x \to \infty$ as $x \to x_1$). The appearance of a singularity implies the unsuitability of the quasicylindrical approximation for describing the flow near the point x_1, and indicates certain special properties of the profiles $u(y, x_1)$, $g(y, x_1)$ at this point.

In the modern asymptotic theories of detached flows /8, 9/ which take into account the interaction between the boundary layer and the outer flow, the point of separation does not

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coincide with the singularity of the boundary layer equations with a given pressure gradient. The flow patterns in the neighbourhood of these points are also different. The presence of a singularity merely indicates the impossibility of the existence of a flow without separation. An identical relationship obviously exists between the singularity of the quasicylindrical equations with a given pressure gradient, and the point of vortex breakdown.

Thus the vortex breakdown is characterized by the appearance on the axis of a stagnation point, although in all the computations of the quasicylindrical approximation /1, 7/ leading to the appearance of a singularity, the velocity on the axis never became zero on approaching the singularity. Therefore the present paper deals only with the velocity profiles where $u > 0$.

We assume that the velocity and circulation profiles $u(y, x_1) = v(y)$, $g(y, x_1) = \Gamma(y)$ are smooth functions of the form normal for the vortex, i.e. $U > 0$ in the whole interval $y > 0$; $\Gamma(y)$ increases monotonically, $U = \alpha + O(y)$, $\Gamma = O(y)$ as $y \to 0$; $\Gamma \to \Gamma_\infty$, $U \to U_\infty$ exponentially as $y \to \infty$. We use the distance from the point x_1: $\xi = \pm(x - x_1)$ as the small parameter, and the upper and lower signs correspond to the expansions from the left and right respectively. The expansion of the velocity over the point x_1 is regular in x and given by the outer solution

$$u = U_\infty \mp \xi u_\infty'(x_1) + {}^1/_2\xi^2 u_\infty''(x_1) + O(\xi^3), \quad y \to \infty .$$

We shall show which conditions are satisfied by the profiles U, Γ in the cases when a regular expansion near the point x_1 is not possible, and construct the expansions near the singularity.

2. We will introduce a stream function ψ so that $u = \partial\psi/\partial y$, $v = \pm\partial\psi/\partial\xi$. Let us write the expansions of the functions near the point x_1 in the following form (the necessity for precisely this representation is justified below):

$$\begin{aligned} \psi &= \psi_0(y) + \xi^{1/2}\varphi_1(y) + \xi\varphi_2(y) + \xi^{3/2}\varphi_3(y) + \ldots \\ g &= \Gamma(y) + \xi^{1/2}g_1(y) - \xi g_2(y) + \xi^{3/2}g_3(y) + \ldots \\ p &= P(y) + \xi^{1/2}p_1(y) - \xi p_2(y) - \xi^{3/2}p_3(y) + \ldots \\ \psi_0 &= \int_0^y U\,dy . \end{aligned} \tag{2.1}$$

Substituting the expansions (2.1) into the quasicylindrical approximation equations (1.1), we arrive at a sequence of boundary value problems forthe ordinary differential equations (p_i, g_i are expressed in terms of φ_i)

$$\begin{aligned} &L(\varphi_1) = \varphi_1'' - R_1\varphi_1 = 0, \quad L(\varphi_2) = {}^1/_2 R_2\varphi_1^2 \mp \Phi_2 \\ &L(\varphi_3) = R_2\varphi_1\varphi_2 - {}^1/_6 R_3\varphi_1^3 \mp \Phi_3, \quad L(\varphi_n) = G_n \\ &\varphi_k(0) = 0, \quad \varphi'_{2k-1} \to 0 \quad \text{as} \quad y \to \infty; \quad k = 1, 2, \ldots \\ &\varphi_2' \to \mp u_\infty'(x_1), \quad \varphi_4' \to {}^1/_2 u_\infty''(x_1), \ldots \\ &I_0 = \frac{\Gamma^2}{2}, \quad I_n = \frac{I'_{n-1}}{U}, \quad R_1 = \frac{U''}{U} - \frac{I_0'}{2y^2U} \\ &R_n = \frac{R'_{n-1}}{U} - \frac{1}{2y^2U^2} I'_{n-1}, \quad \Phi_2 = \frac{2(yU')''}{U} - \frac{\Gamma\Gamma''}{yU^2} \\ &\Phi_3 = \Big\{\frac{2}{3}\Big[\frac{2}{U}(y\varphi_1'')' - \frac{\Gamma g_1''}{yU^2}\Big] - \frac{1}{3}\frac{\Gamma}{y^2U^2}\Big[\varphi_1\Big(\frac{y\Gamma''}{U}\Big)' - \\ &\qquad 2\varphi_1'\Big(\frac{y\Gamma''}{U}\Big)\Big] + \frac{1}{3U}[\Phi_2'\varphi_1 - 2\Phi_2\varphi_1'] - \frac{\Gamma\Gamma''}{yU^3}\varphi_1\Big\} . \end{aligned} \tag{2.2}$$

A prime denotes differentiation with respect to y, and the right-hand sides of G_n contain only φ_k form the previous approximations. Terms containing R_2, R_3, R_n appear because the equations (1.1) are non-linear, and the quantities Φ_2, Φ_3 are given by the dissipative terms of (1.1).

The homogeneous boundary value problem obtained for φ_1 has, in general, no non-trivial solutions. Writing $\varphi_1 = 0$, we arrive at a homogeneous problem for φ_3 with the same operator L, therefore $\varphi_3 = 0$, etc. This if zero is not an eigenvalue of the operator L, only a regular expansion is possible and the function φ_k is defined uniquely for even values of k.

Now let U, Γ be such that a non-trivial solution of the homogeneous equation φ_0, $\varphi_0(0) = 0$, $\varphi_0(\infty) = 0$ exists, with the normalizing condition $\varphi_0'(0) = 1$. We take

$$\theta_0 = \varphi_0 \int_1^y \frac{dy}{\varphi_0^2}$$

as the linearly independent solution, and varying the constants we obtain the general solution for φ_n

$$\varphi_n = -\varphi_0 \int_0^y G_n\theta_0\,dy + \theta_0\int_0^y G_n\varphi_0\,dy + C_n\varphi_0 + D_n\theta_0. \tag{2.3}$$

Requiring that the boundary conditions hold and taking into account the fact that

$$\varphi_0 = y - {}^1/_2 ay^2 + O(y^3), \quad \theta_0 = -1 + 2ay\ln y + \ldots$$

as $y \to 0$, we obtain

$$\varphi_n'(\infty)\varphi_0(\infty) = \int_0^\infty G_n\varphi_0\,dy\,. \tag{2.4}$$

It can be shown that the quantity $\varphi_n'(0)$ is finite, therefore irrespective of the fact that the operator L is obtained from the inviscid approximation and contains only a second-order derivative, the symmetry condition on the axis $\partial u/\partial r = 0$ is satisfied automatically, i.e. there is no need for an additional investigation of the viscous sublayer, as was done in e.g. /7/.

Condition (2.4) (which can also be obtained by applying Green's formula to the boundary value problem) cannot be satisfied in the general case without introducing an irregularity into the expansion.

The equation for the regular term φ_2 is always inhomogeneous due to the presence of the dissipative term Φ_2, and to satisfy condition (2.4) we must introduce into G_2 a free constant through the non-linear terms, i.e. we must construct the expansions in powers of $1/n$ where n is an integer.

However, as we shall show, the appearance of the powers ${}^1/_3, {}^1/_4$, etc. is possible only when additional conditions are met. Therefore the expansion (2.1) in semi-integral powers represents the most general expansion near the singularity. Introduction of the logarithmic terms into the expansion used in analogous cases in the viscous sublayers, does not yield the required result for the operator L.

Let $\varphi_1 = C_1\varphi_0$. Then the condition that (2.4) has a solution will yield the following expression for the second equation:

$$\frac{C_1^2}{2} = \mp\Big(u_\infty'(x_1)\varphi_0(\infty) - \int_0^\infty \Phi_2\varphi_0\,dy\Big)J^{-1}, \quad J = \int_0^\infty R_2\varphi_0^3\,dy\,. \tag{2.5}$$

According to (2.3), $\varphi_2 = \varphi_{20} + C_2\varphi_0$ and the necessary condition for the third equation to have a solution is, that the following condition holds:

$$C_2 = \Big(\pm\int_0^\infty \Phi_3\varphi_0\,dy - \frac{C_1^2}{6}\int_0^\infty R_3\varphi_0^4\,dy - \int_0^\infty R_2\varphi_0^2\varphi_{20}\,dy\Big)J^{-1}.$$

Continuing this process, we shall express the constants C_n in terms of the functions U, Γ and of the velocity expansion coefficients in the external flow.

Equations (2.5) implies that when a specific condition is imposed on the velocity gradient in the external flow, namely

$$u_\infty'(x_1) = \frac{1}{\varphi_0(\infty)}\int_0^\infty \Phi_2\varphi_0\,dy \tag{2.6}$$

then a regular expansion can be constructed irrespective of the existence of the characteristic solution φ_0. Here we have $\varphi_k = 0$ for odd k; $\varphi_2 = \varphi_{20} + C_2\varphi_0$, C_2 are determined from the condition for the equation to be solvable for φ_4, $\varphi_4 = \varphi_{40} + C_4\varphi_0$, etc.

However, in any case the existence of a characteristic solution means that the solution is unstable to small perturbations since the constants in the approximation in question are determined by the condition for the higher-order approximation to be solvable, and hence the change in the boundary conditions in the higher approximation, e.g. $u_\infty'(x_1)$, will influence the magnitude of the given approximation (on C_1). The choice of the sign of C_1 must follow from numerical integration of the quasicylindrical approximation equations and obviously cannot be established from a local analysis.

If condition (2.6) does not hold, expansion (2.1) can be constructed only on one side of the singularity (according to (2.5) the constant C_1 must be imaginary on the other side). Note that the impossibility of continuing the solution past the singularity was discovered in /2, 3/ for singularities in a boundary layer with a given pressure gradient.

3. Expansion (2.1) cannot be constructed if

$$J = 0\,. \tag{3.1}$$

In this case we must construct the expansion in powers of $^1/_3, ^2/_3, \ldots$, although the expressions for the constants will contain, as before, a denominator depending on U, Γ. When the denominator becomes equal to zero, we must change over to an expansion in powers of $^1/_4, ^1/_2, ^3/_4, \ldots$, etc.

Let us consider now the conditions that expansions begin with the power $1/n$ are formulated

$$\psi = \psi_0 + \xi^{1/n}\varphi_1 + \xi^{2/n}\varphi_2 + \ldots \xi^{(n-1)/n}\varphi_{n-1} + \xi\varphi_n + \ldots.$$

We have for all $k < n$ the uniform boundary conditions on φ_k, and the viscous terms do not occur in the right-hand sides of the equations

$$L(\varphi_1) = 0, \quad L(\varphi_2) = {}^1/_2 R_2\varphi_1^2, \quad L(\varphi_3) = R_2\varphi_1\varphi_2 + {}^1/_6\, R_3\varphi_1^3$$
$$L(\varphi_4) = \frac{1}{24} R_4\varphi_1^4 + \frac{1}{2} R_3\varphi_1^2\varphi_2 + \frac{1}{2} R_2\varphi_2^2 + R_2\varphi_1\varphi_3. \tag{3.2}$$

Let a characteristic solution φ_0, $\varphi_1 = C_1\varphi_0$ exist. Then, provided that the condition (3.1) for the second equation to be solvable holds, we obtain $\varphi_2 = C_1^2\varphi_{20} + C_2\varphi_0$. The necessary condition for the third equation to be solvable is, that

$$\int_0^\infty \left(R_2\varphi_0\varphi_{20} + \frac{1}{6} R_3\varphi_0^4\right) dy = 0. \tag{3.3}$$

Then $\varphi_3 = C_1^3\varphi_{30} + 2C_1C_2\varphi_{20} + C_3\varphi_0$.

The condition for the fourth equation to be solvable has the form

$$\int_0^\infty \left(\frac{1}{24} R_4\varphi_0^5 + \frac{1}{2} R_3\varphi_0^3\varphi_{20} - \frac{1}{2} R_2\varphi_{20}^2\varphi_0 + R_2\varphi_0^2\varphi_{30}\right) dy = 0. \tag{3.4}$$

The process can be continued. For example, if φ_0 exists, conditions (3.1), (3.3), (3.4) hold and the condition for the fifth equation of (3.2) to be solvable does not hold, then we have an expansion in powers $^1/_5, ^2/_5, ^3/_5, \ldots$. Note that the number n depends on the form of the profiles U, Γ only, and therefore characterizes the state of the flow at the given point.

Thus in order to have an expansion in powers of $1/n$, φ_0 must exist, and an additional $n-2$ conditions of the type (3.1), (3.3), (3.4) must hold. The expansion obtained will contain $n-2$ arbitrary constants.

4. Let us investigate the possibility of singularities appearing on two different families of profiles.

The two-parameter family of profiles

$$U = 1 - (\alpha - 1)e^{-y}, \quad \Gamma = \gamma\bar{\Gamma} = \gamma(1 - e^{-y}) \tag{4.1}$$

can be regarded as the limiting case of a solution describing a flow in a vortical trace /10/. This family was used in /11/ to model a flow in the linear analysis of the stability of vortex lines. We know that such profiles approximate well the velocity and circulation distributions upstream from the point of vortex breakdown /5/.

The two-parameters family

$$U = \alpha + (1-\alpha)\,4y\,(3 - 4\sqrt{2y} + 3y), \quad \Gamma = \gamma\bar{\Gamma} = 4\gamma y\,(1-y) \tag{4.2}$$
when $y < {}^1/_2$
$U = 1, \ \Gamma = \gamma$ when $y \geqslant {}^1/_2$

was used in /12/ to analyse vortex breakdown using the integral method, and as a boundary condition in the incoming flow upstream of the point of vortex breakdown /13/.

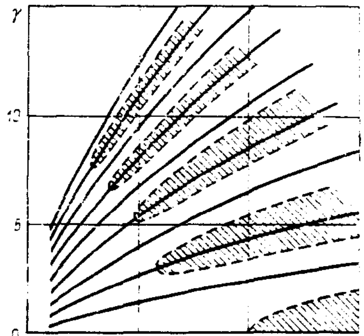

Fig.1

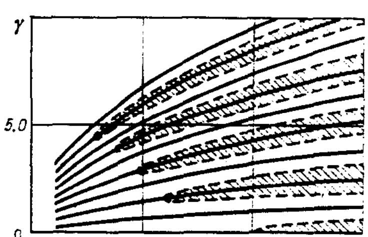

Fig.2

For the families (4.1) (4.2) $\alpha = U(0)$ is the velocity on the axis and $U(\infty) = 1$, $\gamma = \Gamma_\infty$ characterizes the twist intensity. We will compute the functions R_1, R_2, Φ_2 over the distributions (4.1), (4.2). Next we integrate for fixed α the equation $\varphi''_0 - R_1\varphi_0 = 0$ with initial conditions $\varphi_0(0) = 0$, $\varphi_0'(0) = 1$ upwards from the axis $y = 0$, removing the values of the parameter γ at which $\varphi_0'(\infty) = 0$ (for the family (4.2) the integration is carried out over the interval $(0, {}^1/_2)$).

Figs.1 and 2 show the results of the computations for the families of profiles (4.1), (4.2) respectively. The solid lines show the values of the parameters α, γ for which the characteristic solution $\varphi_0(y)$ exists. This implies that a regular expansion near the profiles with such parameter values is in general impossible.

For fixed α an infinite increasing sequence of the values $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ exists for which $\varphi_0'(\infty) = 0$, and for $\gamma = \gamma_n$ the function $\varphi_0'(y)$ vanishes within the interval of integration $n - 1$ times. Using the WKB method /14/ to analyse the distribution of the eigenvalues, we can obtain an asymptotic expression for γ_n which holds as $n \to \infty$

$$\gamma_{n+1} - \gamma_n = \pi\left(\int_0^\infty \left(\frac{\Gamma\Gamma'}{2y^2U^2}\right)^{1/2} dy\right)^{-1}. \tag{4.3}$$

The relation (4.3) and the results of the computations shown in Figs.1 and 2 show that the lines $\gamma_n(\alpha)$ narrow during the limiting passage as $\alpha \to 0$, with fixed $\gamma > 0$, $\gamma_n(0) = 0$.

The value of J in (2.5) is less than zero in the shaded areas in Figs.1, 2; consequently points exist on the lines $\gamma_n(\alpha)$ for even n, at which the expansion begins with the power $1/n$ $(n \geqslant 2)$.

The classification of the flows into supersonic and subsonic /4/ is based on a study of small flow perturbations whose scale is of the order of the vortex radius, i.e. on the lines which are much smaller than the characteristic longitudinal size in the quasicylindrical approximation (the effect of viscosity is insignificant in relation to these scales). Writing the stream function in the form $\psi(x, y) = \psi_0(y) - \delta e^{i\lambda X}\varphi(y)$ where $X = x/\varepsilon$ and passing to the limit as $(\varepsilon \to 0, \delta \to 0)$ in the Navier-Stokes equations, we obtain the following eigenvalue problem:

$$\varphi_{yy} + (\lambda^2/2y - R_1)\,\varphi = 0, \qquad \varphi(0) = 0, \quad \varphi'(\infty) = 0.$$

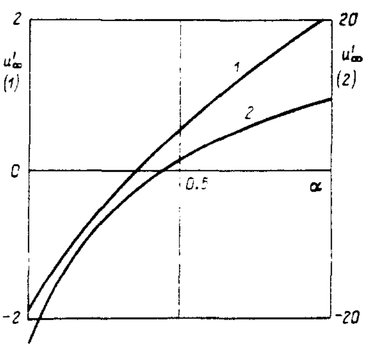

Fig.3

Under the assumptions made above about U, Γ, the problem has an infinite series of real eigenvalues $\lambda_0^2 < \lambda_1^2 < \lambda_2^2 \ldots$. If $\lambda_0^2 < 0$, and therefore if the standing waves can be maintained, the state with U, Γ will be called "subcritical", and "supercritical" when $\lambda_0^2 > 0$ /4/. When the twist increases, the values λ_1^2, λ_2^2 etc. decrease and pass through zero one after the other. Comparing problems (2.2) and (4.1) we find, that every consecutive passage of the eigenvalues λ_k through zero will lead to a singularity in the quasicylindrical approximation. This means that the qausicylindrical approximation generates a singularity only in response to the states U, Γ capable of maintaining the standing waves of infinite length (in the scale X). A singularity with minimum twist and the function φ_0 without any zero within the interval, corresponds to critical flows and can be ordered according to the number of zeros of the function φ_0 within the interval.

Fig.3 shows the distributions $u_\infty'(x_1)$ along the lines of critical state (the lower lines in Figs.1, 2) for which there is no singularity in the critical state. Line 1 corresponds to the family (4.1), and line 2 to (4.2).

We see that a singularity may appear even when the pressure gradient in the outer flow is favourable $u_\infty'(x_1) > 0$, which is not the case for a boundary layer. However, when the values of $u_\infty'(x_1)$ lie above the line in Fig.3, i.e. when the outer flow diverges sufficiently, a singularity cannot appear.

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STATIONARY MODEL OF THE GENERALIZED PRANDTL EQUATIONS AND THE PASSAGE TO THE LIMIT WITH RESPECT TO LONGITUDINAL VISCOSITY IN THE NAVIER-STOKES EQUATIONS*

V.M. SOLOPENKO

It is shown that the generalized Prandtl equations (GPE) represent a limiting case of the Navier-Stokes (NS) equations when the "longitudinal" viscosity tends to zero. An estimate for the neglected terms is obtained and a theorem of existence proved for the GPE. The theorem was established earlier /1/ for the case of homogeneous conditions.

The passage to the limit of the non-steady Euler equations is carried out in /2/ under the assumption that the vorticity vanishes on the solid surfaces. Although the assumption is not physically justified, it enables the integrals over the solid surfaces to be estimated easily.

It is well-known that the use of the Hopf truncation for the NS equations in the inhomogeneous stationary problemof flow, leads to an estimate of the nor of the velocity gradient depending exponentially on viscosity /2, 3/. We note that no such difficulty arises in the case of the non-stationary problem, nor in the Cauchy problem /4, 5/. In the first case the "smoothing" may take place with time, and in the second case there are no boundary effects at all.

The problem of flow with various boundary conditions specified in terms of the stream and Bernoulli functions, free from the above drawbacks, is studied below.

1. Formulation of the problem. The flow takes place within the square $\Omega = (0.1) \times (0,1)$. We denote the segment $x = 0$ by Γ_1 and number the remaining sides $\Gamma_{2,3,4}$ in an anticlockwise direction $\Gamma_{1,3}$ denote the inflow and outflow segments respectively, and $\Gamma_{2,4}$ are rigid walls. Introducing the Bernoulli function $H = p + {}^1/_2 (\psi_y')^2 + {}^1/_2 (\psi_x')^2 + \Pi$ (the notation is standard), we consider the system of equations

$$\begin{aligned}(\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_x' + H_y' &= \psi_y'\Delta\psi + f_2, \quad 0 \leqslant \nu_1 \leqslant \nu_2 \\ (\nu_1\psi_{x^2}'' + \nu_2\psi_{y^2}'')_y' - H_x' &= -\psi_x'\Delta\psi - f_1, \quad \nu_2 > 0\end{aligned} \tag{1.1}$$

where $f_{1,2}$ are the components of the mass force vector. When $\nu_1 = \nu_2 = \nu$, we have the NS equations and when $\nu_1 = 0$, we have the GPE. Eliminating H, we can rewrite (1.1) in the following equivalent form:

$$\nu_1 (\Delta\psi)_{x^2}'' + \nu_2 (\Delta\psi)_{y^2}'' = -\psi_x' (\Delta\psi)_y' + \psi_y' (\Delta\psi)_x' + f_{2x}' - f_{1y}'. \tag{1.2}$$

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